NASA TT F-9729

QUASILINEAR THEORY OF PLASMA OSCILLATIONS

A. A. Vedenov, Ye. P. Velikhov and R. Z. Sagdeyev

Translation of "Kvazilineynaya Teoriya Kolebaniy Plazmy".
Nuclear Fusion (Internat.), Suppl. Pt. II,
pp. 465-475, 1962. "Plasma Physics"
Conference Paper, Salzburg, 1961.

NATIONAL AERONAUTICS AND SPACE ADMINISTRATION
WASHINGTON OCTOBER 1965

QUASILINEAR THEORY OF PLASMA OSCILLATIONS[1]

A. A. Vendenov, Ye. P. Velikhov and R. Z. Sagdeyev

36762

A method of investigating non-equilibrium processes in systems with collective degrees of freedom is developed. This method allows states of non-equilibrium with strongly excited (suprathermal) oscillations to be studied. */465

As applied to plasma, this method is as follows: when the non-equilibrium processes are examined, allowance is made for the influence of a self-consistent oscillatory field on the particle distribution function. The distribution function appears as a sum of slowly and rapidly varying terms. In the equation for the "slow" distribution function f_o, the quadratic mean effect of the oscillations is taken into consideration.

The quasilinear method is used for the study of the problem of the absorption of the energy of finite-amplitude waves in a rarefied plasma. It is shown that, for suprathermal oscillations, damping is considerably less than that obtained from the ordinary linear theory, the magnitudes of the damping decrement being in inverse proportion to the energy density ε of the waves. The expression for the damping decrement γ, valid when $\varepsilon \ll kT$, is of the form

$$\gamma = \gamma_0 [1 + A (\varepsilon/kT) N_D]^{-1}$$

where $A \sim 1$, N_D is the number of particles in a sphere with a radius equal to the Debye radius, and γ_o is the decrement given by linear theory.

The above formula for damping is applicable only in cases where the wave packet is sufficiently "wide": $\Delta v_\phi > (e\phi_o/m)^{\frac{1}{2}}$ (where Δv_ϕ is the spread of wave velocities in the packet and ϕ_o is the amplitude of the potential). In the contrary

[1]Paper No. CN-10/199. Speaker: Ye. P. Velikhov. Discussion (in English) appears on p. 493. Translations of the annotations are included at the end of this volume of the "Proceedings" of the conference.

*/Numbers in the margin indicate pagination of the original foreign text.

limiting case of a monochromatic wave the "damping decrement" depends on the amplitude as $\varepsilon^{-3/4}$. Such dependence occurs, for instance, when oscillations are damped within a shock wave front in a rarefied plasma with no magnetic field present. The results obtained are discussed in connection with diagnostic problems and in connection with the problem of high-frequency heating of a plasma.

Oscillation processes in unstable plasmas not too far away from the boundary of stability are investigated. The stationary noise spectrum in a plasma carrying an electric current under conditions of ion electrostatic instability is found for purposes of illustration. The amplitude of the stationary oscillations is determined. In cases when the plasma is situated in a strong magnetic field and the current flows in the direction of that field, the shape of the oscillation spectrum is given by the expression $\varepsilon_k \sim x^3(1-x^2)^{3/2}$, where ε_k is the spectral density, $x = \omega_k/ks$, $s^2 = T_e/M$.

The limits of applicability of the quasilinear theory just developed, the nonlinear effects of wave interaction, and the question of transition to "turbulent" conditions are discussed.

1. Introduction

Among the most salient features of plasmas are the seven branches of their oscillation spectrum which are usually excited far above the equilibrium thermal level.

At the present time, it is only the theory of small plasma oscillations, based on linearized equations, which has undergone any degree of elaboration. This theory permits one to find the dispersion properties of a plasma for various types of oscillations and the conditions under which the latter increase spontaneously (i.e., the instabilities of the plasma). What it cannot do, however, is give the amplitude attained by the oscillations and indicate how they affect transfer processes in the plasma, a most important matter which arises in dealing with the problem of magnetic thermal insulation of plasmas.

Nonlinear plasma motions, on the other hand, have only been studied for several special cases and under certain simplifying assumptions. One such case is that of steady plane waves, when the only remaining dependence is on a single space coordinate. These particular solutions nonetheless show that even with small but finite amplitudes effects proceed not as predicted by the linearized theory. This applies, for example, to Landau damping, which follows from the linearized theory but actually is not involved in the nonlinear consideration of steady waves in a collisionless plasma. As we know, the reason for this is as follows: the particles responsible for the absorption of waves in a plasma are particles (ions, electrons) which are in resonance with the wave; even with small amplitudes the distribution becomes severly distorted with time due to the reaction of the wave field -- something not taken into account by the linear theory. It is clear, moreover, that the reaction effect plays an equally important role in the buildup of oscillations in an unstable plasma. /466

2. General Formalism of the Quasilinear Theory

Thus, the description of phenomena with small but finite amplitudes requires consideration of the effect of oscillation reaction on the distribution of particles in the velocity space.

Such consideration can be effected within the framework of the "quasilinear" theory which is the subject of this paper. In the quasilinear approximation the particle velocity distribution function is represented as the sum of two terms: the slowly varying term f_0 ($\underline{v}$, t) (which we call the "background") and the rapidly varying term f_1 ($\underline{v}$, t).[1] The slow variation of the "background"

[1] In the linear theory f_0 is considered to be a given function.

as a result of the effect of oscillation reaction on the particles is brought about by the mean square effects of small rapid oscillations. In this sense we have here an analogy with the familiar Van der Pol method of nonlinear mechanics.

It is important to note that the quasilinear approximation does not take into account interaction between the various "harmonics" and "modes". For this reason, the energy balance in the <u>k</u>-th harmonic of the oscillations (<u>k</u> is the wave vector) is determined, as in linear theory, by the equation $d\varepsilon_k/dt = 2\gamma \cdot \{f_o\}\varepsilon_k$, where γ is the imaginary component of the frequency and is a functional of the "background" f_o.

As a very simple illustration let us derive the quasilinear-theoretical equation for the longitudinal Langmuir electron oscillations of a rarefied, completely ionized high-temperature plasma. As we know, the processes in such a plasma can be described by means of a kinetic equation with the self-consistent field E, the effects of collisions between paarticles being neglected,

$$\frac{\partial f}{\partial t} + v\frac{\partial f}{\partial x} + \frac{eE}{m}\frac{\partial f}{\partial v} = 0 \qquad (1a)$$

$$\frac{\partial E}{\partial x} = 4\pi Ne\left(\int f\,dv - 1\right). \qquad (1b)$$

Separating the distribution function into its slowly and rapidly varying terms,

$$f = f_o + f_1,$$

(hence, the average value of the rapidly oscillating term is equal to zero, $\langle f_1 \rangle = 0$, so that $\langle f \rangle = f_o$), and setting

$$f = f_0 + f_1,$$

$$f_1 = \frac{1}{\sqrt{V}} \sum_k (f_k e^{ikx - i\omega_k t} + c.c.)$$

$$E = \frac{1}{\sqrt{V}} \sum_k (E_k e^{ikx - i\omega_k t} + c.c.) \tag{2}$$

we equate the oscillating terms in the right and left sides of (1) to obtain a relationship between f_1 and E,

$$f_k = \frac{e}{m} \frac{1}{i(\omega_k - kv)} \frac{\partial f_0}{\partial v} E_k \tag{3}$$

and the usual expressions for the real and imaginary parts of ω_k which follow from the linear theory,

$$\mathrm{Re}\, \omega_k = \Omega_k \{f_0\} \tag{4a}$$

$$\mathrm{Im}\, \omega_k = \gamma_k \{f_0\}. \tag{4b}$$

The equation for the slowly varying term f_0 of the distribution function is obtained by substituting expression (2) for E and f_1 into (1a) and averaging,

$$\frac{\partial \langle f \rangle}{\partial t} + v \frac{\partial \langle f \rangle}{\partial x} + \left\langle \frac{eE}{m} \frac{\partial (f_0 + f_1)}{\partial v} \right\rangle = 0. \tag{5}$$

The following important point must be made in connection with the discussion to follow: we will assume that the plasma simultaneously contains many waves with different wave vectors and chaotically distributed phases; thus, we will be considering wave packets of such width that it will be possible to neglect the capture of particles by the "potential wells" of individual packet harmonics. In the case of longitudinal Langmuir oscillations we are presently considering, this requires that the spread of the phase velocities ω/k of the waves in the packet exceed substantially the velocity $e\phi_0$ with which a wave-captured particle would move in the "potential well",

$$\Delta\left(\frac{\omega}{k}\right) \gg \left(\frac{e\varphi_0}{m}\right)^{1/2}. \tag{6}$$

If condition (6) is fulfilled with the result that there are no captured particles, we can assume the mean distribution function $\langle f \rangle = f_0$ to be homogeneous in space, so that $\partial\langle f \rangle/\partial x = 0$. Recalling that $\langle E f_0 \rangle = \langle E \rangle f_0 = 0$, we obtain from (5) the following equation for the "background" f_0:

$$\frac{\partial f_0}{\partial t} = \frac{\partial}{\partial v} D \frac{\partial f_0}{\partial v}, \tag{7}$$

where the "coefficient of diffusion in the velocity space" D is proportional to the square of the electric field of the waves,

$$\begin{aligned} D &= \frac{e^2}{m^2} \sum_{kk'} \Big\langle (E_{k'} e^{ik'x - i\omega_{k'}t} + \text{c.c.}) \\ &\qquad \times \left(\frac{E_k}{i(\omega_k - kv)} e^{ikx - i\omega_k t} + \text{c.c.}\right)\Big\rangle \\ &= \frac{e^2}{m^2} 2\pi \sum_k |E_k|^2 \delta(\omega_k - kv). \end{aligned} \tag{8}$$

On the other hand, the rate of change of the wave energy in the spectral interval (k, k + dk) is given by formula (4b), which in the case of long-wave ($kR_D \ll 1$) Langmuir electron oscillations is of the form[1] /467

$$\frac{1}{|E_k|^2} \frac{d|E_k|^2}{dt} = 2\gamma_k \{f_0\} = \frac{\pi\omega^3}{k^2} \left(\frac{\partial F}{\partial v_\parallel}\right)_{v_\parallel = \omega/k} \tag{9}$$

where $F(v_\parallel) = \int f_0 \, dv_\perp$ is the distribution of the electrons over velocities parallel to the direction of wave propagation. The system of equations (7), (8), (9) of the quasilinear theory is a closed one; it describes the reverse effect of the Langmuir oscillations on the particle distribution function. It should be noted from the start that an equation of the type (7) may be meaning-

[1]When $kR_D \ll 1$, $\omega \approx \Omega_k = [\omega_e^2 + 3(T/m)k^2]^{1/2}$, $\omega_e^2 = 4\pi Ne^2/m$.

fully considered only if the amplitude of the oscillations is considerably larger than the amplitude of thermal plasma noises in the corresponding portion of the spectrum. This is due to the fact that, as was shown in [1], consideration of thermal noises merely results in a change of the quantity subsumed by the logarithmic symbol in the Coulombic collision term (which is tantamount to exceeding the limits of accuracy).

Where necessary, collisions are taken into account only to within logarithmic accuracy. For this reason, the effect of thermal noises whose energy density is on the order of magnitude of NT/N_D, where N_D is the number of particles in a sphere of Debye radius R_D, is neglected. In other words, we limit ourselves to the study of the "suprathermal" oscillations, whose energy density $\varepsilon \gg NT/N_D$.

It is necessary also to note the fact that the applicability of the quasi-linear theory equations is limited to cases where the oscillatory increment (or decrement) is considerably less than the frequency of the oscillations; if, on the other hand, this condition is not fulfilled, separation of the distribution function into rapidly and slowly varying terms is impossible and equations of the type (7), (8), (9) are invalid.

It is clear from the form of equation (7) for the mean particle distribution function f_0 that the excitation of collective degrees of freedom (waves) in a plasma involves the appearance of additional diffusion in the velocity space in addition to the usual "collisional" diffusion. It is interesting that the ratio of the coefficient of "wave diffusion" D to the "collisional diffusion" coefficient $D_0 \sim Ne^4/mv$ (where v is the average thermal velocity of the electrons),

$$\frac{D}{D_0} \sim \frac{\omega_e E^2/mN}{Ne^4/mv} \sim \frac{E^2/N}{e^2/R_D} \sim \frac{E^2}{NT} \cdot N_D, \tag{10}$$

(where $N_D \sim NR_D^3$ is the number of particles in a sphere of Debye radius R_D) is equal in order of magnitude to the ratio of the wave energy (per particle E^2/N to the energy of electrostatic interaction (likewise per particle) e^2/R_D.[1]) Thus, the effects of particle interaction with excited collective degrees of freedom are inversely proportional to system non-idealness.

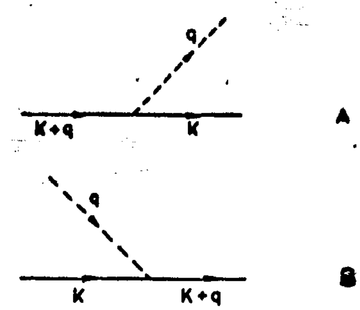

Fig. 1. Emission (A) and absorption (B) of a wave by a particle.

In order to investigate the physical meaning of the quasilinear theory and to generalize the resulting equations (7)-(9) to include the case of an arbitrary system with strongly excited collective degrees of freedom, we will consider such a system as comprised of two gases -- a particle gas and a wave gas -- and write the equations for the conservation of the particles and waves in the phase space.

Let us consider first the balance equation for the number of particles in the velocity space (assuming that the system is homogeneous in the coordinate space). Since the collective modes are strongly excited (so that the density of the wave gas is considerably higher than its thermodynamic equilibrium value), we can only take into account the particle-wave interaction processes

---

[1]This is the familiar "Debye correction" introduced into the thermodynamic potentials of Coulombic systems.

(and in the first approximation at that). The basic process whose effect we must consider is the "first-order" process[1], i.e., the emission or absorption of a wave q by a particle k (Fig. 1).

The matrix elements of processes 1A, 1B are proportional to $\sqrt{N_q}$ and $\sqrt{N_q+1}$, respectively, where N_q is the number of waves per unit volume of the phase space;[2] in our case, however, $N_q \gg 1$, so that the probabilities of the two processes are equal and proportional to N_q:

$$W(k, q) = N_q W_{k,k+q} \delta(\varepsilon_i - \varepsilon_f)$$
$$(W_{k,k+q} = W_{k+q,k}). \tag{11}$$

As a result of the emission or absorption of waves, a particle experiences a change in momentum and changes its position in the phase space; the change in the number of particles at the point k of the phase space consists of "departure terms" due to absorption

$$-\int dq f_k N_q W_{k,k+q} \delta(\varepsilon_k + \hbar\omega_q - \varepsilon_{k+q}) \tag{12a}$$

and emission /468

$$-\int dq f_k N_q W_{k,k-q} \delta(\varepsilon_k - \hbar\omega_q - \varepsilon_{k-q}) \tag{12b}$$

[1]The very fact of describing excited system states by means of two gases, the particle gas and the wave gas, implies that the interaction between the two is slight. We can therefore assume that higher-order processes play a lesser role than first order particle-wave interaction processes. Specifically, for such systems with Coulombic interaction as that of a rarefied plasma or an ultra-dense electron plasma, the interaction between particles and waves is electro-dynamic and proportional to the particle charge; the weakness of particle-wave interaction in this case has to do with the smallness of the parameter $(e^2/\langle\varepsilon\rangle)/r$ (where $\langle\varepsilon\rangle$ is the average particle energy and r is the average distance between the particles), which is proportional to the square of the charge.

[2]We assume that the waves are Bose-Einstein statistical.

and of analogous "arrival terms"due to absorption

$$+\int \mathrm{d}q\, f_{k-q} N_q W_{k-q,k}\, \delta(\varepsilon_{k-q} + \hbar\omega_q - \varepsilon_k) \tag{12c}$$

and emission

$$+\int \mathrm{d}q\, f_{k+q} N_q W_{k+q,k}\, \delta(\varepsilon_{k+q} - \hbar\omega_q - \varepsilon_k), \tag{12d}$$

where f_k is the distribution function (the diagonal part of the density matrix at k -- the particle representation), ε_k is the kinetic energy of a particle with the wave vector $\underline{k}$, ε_q is the energy of the wave q.

Summing the contributions of the various processes (12), we obtain the following equation for the distribution function f_k:

$$\frac{\partial f_k}{\partial t} = \int \mathrm{d}q\, N_q (\Psi_{k+q,q} - \Psi_{k,q}) \tag{13a}$$

$$\Psi_{k,q} = (f_k - f_{k-q}) W_{k,k-q}\, \delta(\varepsilon_k - \varepsilon_{k-q} - \hbar\omega_q). \tag{13b}$$

In a similar way we obtain the equation for the wave distribution function N_q. N_q varies as a result of the birth and destruction of waves by particles[1], so that in the spatially homogeneous case ($\partial/\partial x = 0$)

$$\begin{aligned} \frac{\partial N_q}{\partial t} = & -\int N_q W_{k,k+q} f_k\, \delta(\varepsilon_k + \hbar\omega_q - \varepsilon_{k+q})\, \mathrm{d}k \\ & + \int N_q W_{k+q,k} f_{k+q}\, \delta(\varepsilon_{k+q} - \hbar\omega_q - \varepsilon_k)\, \mathrm{d}k \\ = & N_q \int \Psi_{k+q,q}\, \mathrm{d}k. \end{aligned} \tag{14}$$

Equations (13)-(14) describe non-equilibrium processes in systems with strongly excited collective degrees of freedom (when there are no external for-

[1] Once again, we take into account first-order processes only and assume that $N_q \gg 1$.

ces $\underline{F}$ and no spatial gradients[1]).

If the relative change in the momentum of a particle associated with the birth or destruction of a wave is small,

$$\frac{q}{k} \ll 1 \tag{15}$$

then particle balance equation (13) assumes the Fokker-Planck form. Indeed, expanding the function Ψ and the difference $\Psi_{k+q,q} - \Psi_{k,q}$ over q in (13), we obtain

$$\Psi_{k,q} = q\frac{\partial f_k}{\partial k} W_{k,k-q}\,\delta\left(\frac{\hbar kq}{m} - \hbar\omega_q\right) \tag{16}$$

and

$$\frac{\partial f_k}{\partial t} = \int \mathrm{d}q\, N_q \cdot q \frac{\partial}{\partial k}\left\{W_{k,k-q}\,\delta\left(\frac{\hbar k}{m} q - \hbar\omega_q\right) q \frac{\partial f_k}{\partial k}\right\}, \tag{17}$$

i.e., the Fokker-Planck equation. In this same approximation the equation for N_q can be written as

$$\frac{\partial N_q}{\partial t} = N_q \int \mathrm{d}k\, W_{k,k-q}\,\delta\left(\frac{\hbar k}{m} - \hbar\omega_q\right) q \frac{\partial f_k}{\partial k}. \tag{18}$$

Specifically, for a system of particles with Coulombic interaction (a plasma) the relative change in the momentum of an electron associated with the birth (absorption) of a quantum of Langmuir oscillations does not exceed $\hbar\omega_p/\langle\varepsilon\rangle$ and is small for rarefied and ultra-dense plasmas, so that equations (17)-(18) apply here; for a rarefied plasma

$$W_{k,k-q} = 4\pi^2 e^2 \frac{\hbar\omega_q}{q^2}$$

---

[1]In order to extend the argument to cover the case where $F \neq 0$; $\partial/\partial x \neq 0$ it is sufficient to replace the partial derivatives $\partial f/\partial t$ by the total derivatives $df/dt = \partial f/\partial t + [\mathcal{H} f]$ in the left sides of equations (13a) and (14).

and (17) coincides with previously derived equation (7), while (18) becomes the usual formula of the linear theory for the increment (decrement) of waves in a plasma (9).

In the case of an arbitrary system with excited collective modes, the particle balance equation is not of the Fokker-Planck form; in the present study we shall be concerned only with the problems of quasilinear plasma theory, and will thus make use of equations of the type (17)-(18) [or (7), (8), (9)].

The technique used to deduce the system of equations of the quasilinear theory (7), (8), (9) can be used to obtain analogous equations for a plasma in a magnetic field (e.g., see [2]). We will not consider the general case here, however, limiting ourselves to a discussion of some actual effects: the development of oscillations in an unstable plasma, the absorption of finite-amplitude waves in a plasma, etc.

3. Development of Instability

In this section we consider the problem of the development of instability in a rarefied plasma within the framework of the quasilinear theory. We shall limit our discussion to instabilities on the Langmuir branch of plasma oscillations, assuming for simplicity that the problem is one-dimensional (the distribution function depends solely on the projection of the particle velocity on a single chosen direction, the Langmuir oscillations occurring in this same direction).[1]

Let us assume that at the initial instant the particle distribution $f(0, v)$ in the velocity space is of the form shown in Fig. 2A; Langmuir oscillations

[1]Such a situation arises, for example, in the presence of a strong magnetic field which sets this direction apart from the rest.

then start to build up in the region where the derivative $\partial f(0, v)/\partial v$ is positive; the spectral density of these oscillations increases [see (9)] as

$$\frac{\partial |E_k|^2}{\partial t} = |E_k|^2 \frac{\pi \omega^2}{k^2} \frac{\partial f}{\partial v}\Big]_{v=\omega/k}. \tag{19}$$

/469

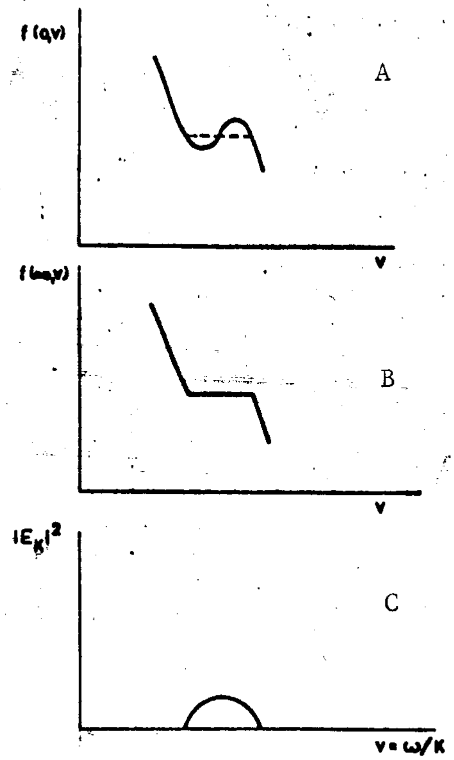

Fig. 2. Formation of a "plateau" in the particle distribution function (A, B) and the spectrum of oscillations as $t \to \infty$ (C).

The appearance of suprathermal noises in the plasma leads in turn to the diffusion of particles in the velocity space,

$$\frac{\partial f}{\partial t} = \frac{\partial}{\partial v} D \frac{\partial f}{\partial v}. \tag{20}$$

The coefficient of diffusion $D(t, v)$ is related in this case to the square of the oscillatory field by the expression

$$D = 2\pi \frac{e^2}{m^2} \int |E_k|^2 \delta(\omega - kv) \frac{dk}{2\pi} = \frac{e^2}{m^2} |E_k|^2 \frac{1}{v}, \tag{21}$$

since in this case involving long waves, the frequency ω coincides with the plasma frequency $(4\pi Ne^2 m)^{1/2}$, and the velocity of the resonance particles is therefore inversely proportional to the wave vector, $v = \omega/k$.

As a result of diffusion, the initial particle distribution smooths out and a "plateau" (Fig. 2B) [3] appears in the velocity interval $v_1 < v < v_2$. At the same time, a steady-state[1] spectrum of suprathermal noises arises in the interval of wave numbers $\omega/v_2 < k < \omega/v_1$. The shape of this steady-state spectrum and the spectral density may be determined as follows.

Integrating (19) over time and neglecting the thermal noise energy in comparison with the energy of the oscillations which are building up, we find the spectral density $|E_k|^2(t)$ of the suprathermal noises,

$$|E_k|^2(t) = \frac{\pi\omega^3}{k^3} \int_0^t dt \, |E_k|^2 \frac{\partial f}{\partial v}. \tag{22}$$

On the other hand, integrating (20) over time and over the velocity v in the interval from v_1 to v, applying (21), and noting that $D\partial f/\partial v = 0$ when $v = v_1$, we obtain

$$\int_{v_1}^{v} [f(t, v') - f(0, v')] \, dv' = \frac{e^2}{m^2} \frac{1}{v} \int_0^t |E_k|^2 \frac{\partial f}{\partial v} \, dt. \tag{23}$$

Comparing (22) with (23), we find the spectral density of the noises,

$$|E_k|^2(t) = \frac{\pi m^2}{e^2} \omega v^3 \int_{v_1}^{v} [f(t, v') - f(0, v')] \, dv'. \tag{24}$$

---

[1]The derivative of the distribution function with respect to velocity vanishes in the region of the plateau, so that the oscillations neither build up nor are damped out.

Upon termination of the process of smoothing of the distribution function, when a plateau $f(\infty, v)$ = const is established in the range $v_1 < v < v_2$, the steady-state spectrum of suprathermal Langmuir oscillations we have been seeking is fully determined by the initial and final distribution functions,

$$|E_k|^2(\infty) = \frac{\pi m^2}{e^2} \omega v^2 \int_{v_1}^{v} [f(\infty, v') - f(0, v')]\, dv'. \tag{25}$$

(the shape of this spectrum is shown in Fig. 2A). By (25), the spectral density of the noises vanishes at the points $k_1 = \omega/v_1$ and $k_2 = \omega/v_2$[1] and its detailed dependence on k is determined by the actual shape of the particle velocity distribution function.

The plasma with a "plateau" in its electron distribution which arises as a result of instability and diffusion has the property that externally excited longitudinal Langmuir oscillations propagate in it without being damped, provided their phase velocity v_f lies within the interval of the plateau $v_1 < v_f < v_2$.

The energy density of the suprathermal noises established by the time that the diffusion process terminates is equal in order of magnitude to

$$E^2 = \int 2|E_k|^2 dk (2\pi)^{-1} \sim \delta n (m v_2^2 - m v_1^2), \tag{26}$$

where δn is the density of that portion of the electrons which diffuse in the velocity space as a result of the emission and absorption of collective Langmuir oscillations and whose kinetic energy is gradually altered thereby. /470

[1]The integral in the right side of (25) vanishes for $v = v_2$ by virtue of the law of conservation of the number of particles:

$$\int_{v_1}^{v_2} f(0, v')\, dv' = \int_{v_1}^{v_2} f(\infty, v')\, dv'.$$

We see, therefore, that the quasilinear theory just presented informs us about the state into which the initially unstable plasma passes as a result of the development of kinetic instability, as well as about the particle distribution function and spectrum of collective oscillations in the final state. It turns out that in a rarefied plasma the relaxation process in the velocity space breaks down into two stages: at first the particle distribution function $f(v)$ smooths out rapidly near the region where there was a positive derivative $\partial f/\partial v$; only later, and much more slowly, does the distribution function tend toward a thermodynamic equilibrium function. It is precisely the first stage, i.e., the establishment of a "plateau" in the distribution function and the appearance of suprathermal noises, which the quasilinear theory describes. If the "plateau" happens to be narrow, system (19)-(21) describing this process can be reduced to a single equation for the coefficient of diffusion D,

$$\frac{\partial D}{\partial t} = D\frac{\partial^2 D}{\partial v^2} + \Phi \cdot D \tag{27}$$

where the function Φ depends solely on the velocity v and coincides to within a factor with the derivative $\partial f/\partial v$ at the initial instant,

$$\Phi(v) = \pi\omega v^3 \partial f(0, v)/\partial v. \tag{28}$$

We note that the steady-state solution of equation (27), which can be obtained by setting $\partial/\partial t = 0$, leads us (by way of (28)) to the formula for the spectral density of the suprathermal noise (25).

Using (27) it is possible to investigate the entire process of development of oscillations in an initially unstable plasma and of the appearance of a plateau in the particle distribution function $f(v)$. The duration τ of the entire process is of the order of magnitude

$$\tau \approx \frac{(v_2 - v_1)^2}{D(\infty)} \sim \left(\frac{v_2 - v_1}{\sqrt{T_e/m}}\right)^2 \cdot \frac{m}{8\pi} \frac{T/m}{(v_2 + v_1)^4} \frac{1}{\omega_e}. \tag{29}$$

4. <u>Absorption of Waves in a Plasma</u>

As we know, the linear theory of small oscillations of a rarefied plasma predicts "collisionless damping" of waves propagating in the plasma. A typical example of such "collisionless damping" is the reduction of the amplitude of longitudinal Langmuir electron waves excited on the boundary of the plasma by an external electric field of frequency $\omega > \omega_e$ which propagate into the interior of the plasma in a direction perpendicular to its boundary. In the case of long-wave oscillations ($kR_D \ll 1$), which are the only ones we shall consider, the reduction of the amplitude of a wave as it penetrates into the plasma is given by[1]

$$\frac{|E_k|^{2\prime}}{|E_k|^2} = \frac{\pi}{3} \frac{\omega_e^4}{k^3} \frac{m}{T} \left(\frac{\partial f}{\partial v}\right)_{v=\omega/k} \tag{30}$$

where ω_e is the plasma frequency, k is the wave vector, and $f(v)$ is the electron distribution function with respect to the velocity component parallel to

---

[1]This expression is valid at distances from the boundary which are in excess of several wavelengths. The effects in the boundary region, where (30) does not apply, do not concern us here. We note that (30) follows from the equation

$$\frac{\partial N_q}{\partial t} + [\mathcal{H} N_q] = N_q \int dk\, W_{k,k-q} \delta\left(\frac{\hbar k q}{m} - \hbar \omega_q\right) q \frac{\partial f_k}{\partial q}$$

if we recall that in the case in question, when $\partial/\partial t = 0$,

$$W = 4\pi^2 \frac{\hbar \omega}{q^2},$$

$$[\mathcal{H} N_q] = \frac{\partial \mathcal{H}_q}{\partial q} \cdot \frac{\partial N_q}{\partial x} - \frac{\partial \mathcal{H}_q}{\partial x} \cdot \frac{\partial N_q}{\partial q} \approx 3 \frac{k}{\omega} \frac{T}{m} \frac{\partial N_q}{\partial x}.$$

the direction of wave propagation. The "prime" denotes differentiation with respect to the x-coordinate.

Thus, the energy of a packet of waves of infinitely small amplitude and different but close frequencies (and wave vectors) diminishes exponentially with distance from the boundary; the spatial decrement of this damping is given by formula (30) with $f = (2 \pi T/m)^{-1/2} \exp(-mv^2/2T)$ (Fig. 3A).

If, on the other hand, we consider the propagation of waves of small but finite amplitude, their damping will be seen to differ radically. Indeed, the quasilinear-theoretical equation for the mean electron distribution function f

$$v \frac{\partial f}{\partial x} = \frac{\partial}{\partial v} D \frac{\partial f}{\partial v} \tag{31a}$$

$$D = 2\pi \frac{e^2}{m^2} \sum_k |E_k|^2 \delta(\omega - kv) = \frac{e^2}{m^2} \frac{|E_k|^2}{v} \qquad (31b)^1$$

implies that the diffusion of particles in the velocity space accompanied by emission and absorption of Langmuir oscillations sharply reduces the derivative $\partial f/\partial v$, i.e., by formula (30), the damping of waves as they pass through the plasma. In the velocity range of the resonance particles the solution of system (30)-(31) is

$$\begin{gathered} f(v,x) = \text{const.} \\ \frac{\partial}{\partial x} |E_k|^2 = 0, \end{gathered} \tag{32}$$

i.e., the waves are not damped at all.

---

[1]The wave vector k and the velocity v of a resonance particle are unambiguously related in this case, $v = \omega/k$. Expression (31b) presumes that a generator which produces a wave at the boundary does not at the same time produce free-flying resonance particles, i.e., it presupposes a certain boundary-value condition. In the contrary case, in the presence of free-flying resonance particles, the results are not qualitatively altered.

In actual fact, the damping of the waves is of small, but nevertheless finite magnitude; this is due to the fact that collisions between particles tend to restore dynamic equilibrium (render the electron distribution function more Maxwellian), i.e., to make the derivative $\partial f/\partial v$ negative (Fig. 1C). In order to find the magnitude of the associated weak wave damping, it is necessary

/471

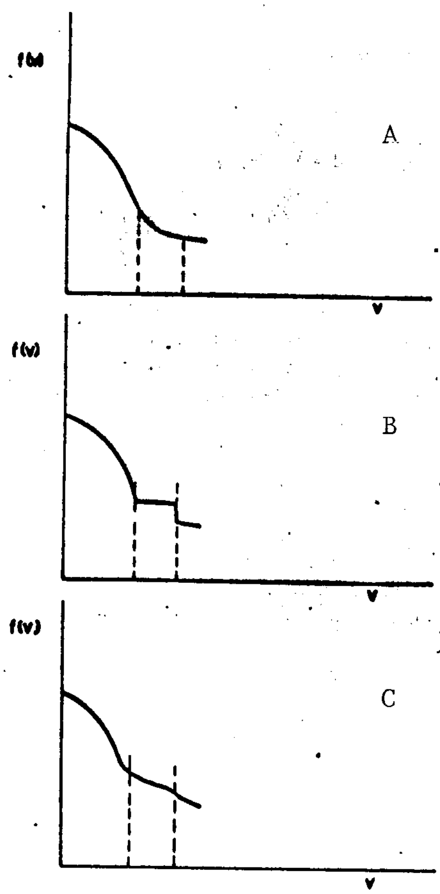

Fig. 3. Distortion of the particle distribution function $f(v)$ under the influence of a Langmuir wave packet.

to introduce a collision term into equation (2) for the mean distribution function f. In the case of long waves we have been considering, the collision term is of the form

$$\mathrm{St}\, f = \frac{\partial}{\partial v} \nu_s \left(v f + \frac{T}{m} \frac{\partial f}{\partial v} \right), \qquad (33)$$

where the "collision frequency" ν_s is equal to

$$\nu_s = S/v^3, \qquad (34)$$

where the order of magnitude of S is equal to $\omega_e{}^4/N$.

With due allowance for collisions between particles, the equation of the plasma electron distribution function becomes

$$v \frac{\partial f}{\partial x} = \frac{\partial}{\partial v} D \frac{\partial f}{\partial v} + \mathrm{St}\, f, \qquad (35)$$

where St f is given by expression (33), and D by formula (31b).

If the amplitude of the waves in the packet is not overly small, so that the inequality

$$\frac{E^2}{4\pi N T} \gg \frac{1}{(N_D)^{1/3}}, \qquad (36)$$

is fulfilled (where $E^2/4\pi$ is the wave energy density), then, as may be shown by estimation, the left side of (35) can be neglected. The derivative of the distribution function, by (35), then becomes

$$\frac{\partial f}{\partial v} = -\nu_s \frac{v f}{D}. \qquad (37)^1$$

If the packet is not too wide, $\Delta(\omega/k) \ll \sqrt{T/m}$, then, as we see from Fig. 3, the distribution function (but not its derivative) changes but slightly under the influence of the waves, so that in the right side of (37) we can replace f by the Maxwellian distribution function $f_M = (m/2\ T)^{1/2} \exp(-mv^2/2T)$, and,

[1]Since in the velocity interval in which we are interested $(T/m)\cdot(\partial f/\partial v)$ is negligibly small in comparison with vf.

substituting in the resulting value of the derivative $\partial f/\partial v = -\nu_s v f_M/D$ into (35), we find that

$$\frac{|E_k|^{2\prime}}{|E_k|^2} = -\frac{\pi}{3}\frac{\omega_e^4}{k^3}\frac{m}{T}\nu_s\frac{v f_M}{D} \tag{38}$$

or, taking into account (31b), we arrive at the equation giving the spatial form of $|E_k|^2$,

$$\frac{\partial |E_k|^2}{\partial x} = -\frac{\pi}{3}\frac{\omega_e^4}{k^3}v^3\frac{m^2}{T^2}\nu_s\frac{f_M}{e^2}. \tag{39}$$

(39) implies that the energy of the wave packet diminishes linearly with distance from the boundary according to the law (Fig. 4)

$$\frac{|E_k|^2(x)}{|E_k|^2(0)} = 1 - \frac{x}{L}, \tag{40}$$

where the characteristic length L (along which the waves excited at the boundary are damped) is directly proportional to the energy of the waves at the

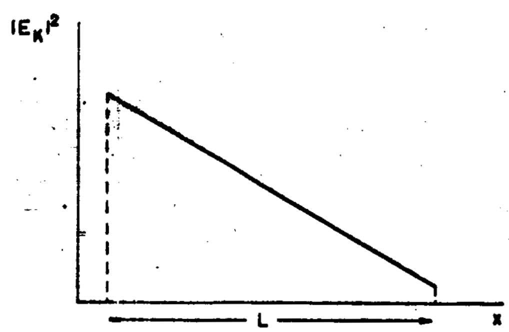

Fig. 4. Damping of Langmuir waves in a plasma.

boundary; its order of magnitude is

$$L \sim k^{-1}\frac{v^2}{T/m}\frac{NT}{E^2}\frac{1}{N_D}, \tag{41}$$

i.e., it exceeds considerably the damping length in the linear theory provided that the wave energy exceeds the energy of the thermal noises, $E^2/NT \gg 1/N_D$ (we recall that in the case under discussion $E^2/NT \gg 1$ $(N_D)^{1/2}$).

To conclude this section we note that the reduced absorption of waves of finite amplitude in a rarefied plasma (see also [2]) should be taken into account in computing plasma heating in a high-frequency field.

5. Plasma in a Constant Electric Field /472

In the present section we shall be concerned with the quasilinear theory of a plasma situated within a constant electric field E. Upon actuation of the field E, a current $j = \sigma E$ flows through the plasma; the plasma resistivity σ^{-1} consists of two terms: the first is determined by collisions between particles and the second by the interaction of charge carriers with fluctuating electric fields.

As the electric field E increases, the mean directed electron velocity v_a increases. At some critical velocity v_c the state of the plasma becomes unstable as low-frequency oscillations (ionic sound) begin to build up in it [3]. The amplitude and spectrum of these oscillations are determined by the balance prevailing between the flux of energy to the oscillations from the electrons moving in the constant electric field E and the flux of energy dissipated in the oscillations. The energy dissipation may be determined by two processes: the transfer of energy over the oscillatory spectrum from certain modes to certain other modes (transfer between collective degrees of freedom) and direct transformation of the oscillatory energy into heat during particle collisions. The characteristic time of energy transfer over the spectrum tends to infinity as the oscillatory amplitude tends to zero, while the characteristic energy dissipation time for individual particle collisions remains constant as this same limit is approached. For this reason, with small super-criticality (i.e., with v_c slightly exceeded) the shape of the oscillatory spectrum and the level of the oscillations are determined by the balance between the energy contributed by the moving electrons and the energy dissipated during particle pair collisions; they can be found with the aid of the quasilinear-theoretical formalism developed in Section 1.

For simplicity, let us consider a partially ionized plasma in which the dominant role is played by collisions between charged and neutral particles. We limit ourselves to the case where the collision frequency ν_s is considerably less than the oscillatory frequency ω, so that neutral particles do not take part in the oscillations. We will consider a plasma situated in a strong magnetic field, so that for the ionic-acoustic branch the oscillatory wavelength is considerably larger than the Larmor radius of the ions.

Dispersion equation (42) for ionic oscillations in a strong magnetic field can be obtained by expanding the kinetic equation for the rapidly oscillating term of the distribution function in the ratio of the characteristic oscillatory frequency to the ionic cyclotron frequency,

$$k^2 \equiv \omega_{pe}^2 \int \frac{F_e'(v_\parallel)(v_\parallel - \omega/k_\parallel)\,dv_\parallel}{(v_\parallel - \omega/k_\parallel)^2 + \nu_e^2/k_\parallel^2} + \omega_{pi}^2 \int \frac{F_i'(v_\parallel)(v_\parallel - \omega/k_\parallel)\,dv_\parallel}{(v_\parallel - \omega/k_\parallel)^2 + \nu_i^2/k^2}. \quad (42)$$

The energy flux balance condition (43) is of the form

$$0 = \omega_{pe}^2 \int \frac{F_e'(v_\parallel)\,\nu_c/k_\parallel\,dv_\parallel}{(v_\parallel - \omega/k_\parallel)^2 + \nu_c^2/k_\parallel^2} + \omega_{pi}^2 \int \frac{F_i(v_\parallel)\,\nu_i/k_\parallel\,dv_\parallel}{(v_\parallel - \omega/k_\parallel)^2 + \nu_i^2/k_\parallel^2},$$
$$\omega^2_{pe} \equiv \omega_c^2 = 4\pi Ne^2/m;\ \omega_{pi}^2 \equiv \omega_i^2 = 4\pi Ne^2/M. \quad (43)$$

In equations (42)-(43)

$$F_{e,i}(v_\parallel) = 2\pi \int f_{e,i}(\varepsilon_\perp, v_\parallel)\,d\varepsilon_\perp$$

represents the steady-state portion of the electron (ion) distribution function with respect to the velocity $v_{\parallel}$ parallel to the direction of the constant magnetic field,[1] and $k_{\parallel}$ is the projection of the wave vector on this direction.

As we know, ionic-acoustic oscillations exist in a plasma only if the electron temperature T_e exceeds considerably the ion temperature T_i; in this case equations (42) and (43) become

$$k^2 = \omega_i^2\left(\frac{k_{\parallel}^2}{\omega^2} - \frac{1}{c^2}\right) \tag{42'}$$

$$F_e'(v_{\parallel}) = \frac{2}{\pi}\frac{m}{M}\nu_i\frac{1}{v_{\parallel}^2\omega(k, v_{\parallel})}. \tag{43'}$$

Here

$$c^2 = -\left\{\frac{M}{m}\int\frac{F_e'(v_{\parallel})}{v_{\parallel} - \omega/k_{\parallel}}dv_{\parallel}\right\}^{-1} \sim \frac{T_e}{M}$$

denotes the velocity of the ionic sound in the low-frequency range ($\omega \to ck_{\parallel}$ as $\omega \to 0$). Using (42') to find the frequency $\omega(k, v_{\parallel})$ and substituting it into (43') we obtain the steady-state condition for a wave directed at an angle θ = arc cos $k_{\parallel}/k$ to the magnetic field,

$$F_e'(v_{\parallel}) = \frac{2}{\pi}\frac{m}{M}\nu_i\frac{1}{v_{\parallel}^2\omega_i\cos\vartheta\sqrt{1 - v_{\parallel}^2/c^2}}. \tag{44}$$

The right side of (44) increases monotonically with increasing θ; hence, if the steady state condition

$$F_e'(v_{\parallel}) = \frac{2}{\pi}\frac{m}{M}\nu_i\frac{1}{v_{\parallel}^2\omega_i\sqrt{1 - v_{\parallel}^2/c^2}}, \tag{45}$$

if fulfilled for all purely longitudinal waves (θ = 0), then all of the "oblique"

[1]The magnetic field of the current flowing in the plasma is considered to be negligibly small in all cases.

waves ($\theta > 0$) are damped out, i.e., in the case of these waves, according to (44), the energy dissipation due to collisions between ions and neutral particles exceeds the influx of energy from the resonance electrons. Thus, in the steady-state the plasma contains only waves directed along the magnetic field.

The form of the distribution function for the electrons in resonance with the steady-state ionic-acoustic noise background can be determined from equation (45); the resulting expression for the distribution function is valid within a limited velocity interval, i.e., where the electron drift induced by the constant electric field maintains the steady-state level of the oscillations.

According to the linear theory, instability arises in that portion of the velocity space where $F_e'(v_{\parallel})$ exceeds the right side of (45).

/473

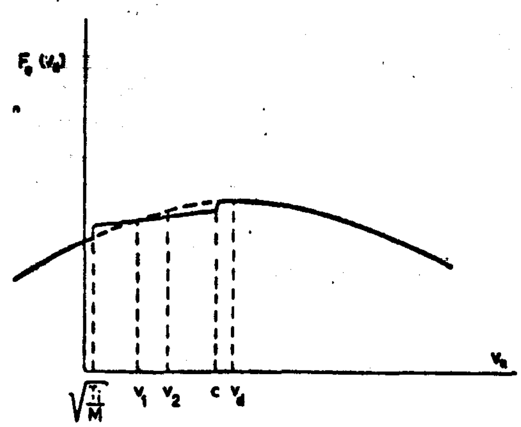

Fig. 5. Function $F_e(v_{\parallel})$ of electron distribution in an electric field.

Fig. 5 shows the "initial" region of instability 1-2. However, as we saw in Section 1, in a time on the order of several oscillatory periods such an instability extends over the entire spectrum of ionic-acoustic oscillations whose phase velocity $\omega/k_{\parallel}$ lies between the mean thermal velocity of the ions

$(T/M)^{\frac{1}{2}}$ [1]) and the velocity $c \sim (T_e/M)^{\frac{1}{2}}$. In this region the function of electron distribution with respect to longitudinal velocities is distorted and assumes the form shown in Fig. 5.

The oscillatory spectrum in this region can be obtained on the basis of the following considerations.

In accordance with the general formalism of the quasilinear theory (Section 1), consideration of the reaction effect of the oscillations on the electrons leads to an equation for the steady-state distribution function $F_e(v_{\parallel})$,

$$\frac{e}{m} E \frac{\partial F_e}{\partial v_{\parallel}} + \frac{\omega_e^2}{m} \frac{\partial}{\partial v_{\parallel}} \frac{\varepsilon_k/N}{v_{\parallel}(1-v_{\parallel}^2/c^2)} \frac{\partial F_e}{\partial v_{\parallel}} = 0, \tag{46}$$

where $\varepsilon_k = E_k^2/4\pi$, $(k = \omega/v_{\parallel})$ is the spectral density of the electrostatic energy. The first term in (46) describes the variation of the electron distribution under the action of the constant electric field E, while the second term is due to the reaction effect of the induced waves on the resonance particles.

Integrating (46), we obtain an expression for the spectral density,

$$\frac{\varepsilon_k/R_D}{NT_e} \frac{v_e}{n} \left(\frac{M}{m}\right)^{3/2} = x^3(1-x^2)^{3/2} \sqrt{\frac{T_e}{m}} \times \left\{\int \frac{v_a}{c} F_e \, dv_{\parallel} + c'\right\}, \tag{47}$$

where the "electron Debye radius" $R_D = \sqrt{T_e/m}\,/\omega_e$, $x \equiv v_{\parallel}/c$.

The integration constant in (47) may be found by imposing the requirement of a minimum total energy in the oscillatory spectrum. This minimality requirement consists in the following. In that region of the velocity space where condition (44) is fulfilled, ionic-acoustic waves (e.g., from some exter-

---

[1] Strong resonance absorption of oscillations by ions which exceeds the electron-induced buildup by a factor of over $\sqrt{M/m}$ begins to occur in the neighborhood of the ionic thermal velocity.

nal source) can propagate without damping. If the spectrum of these waves is such that

$$\frac{\varepsilon_k}{v_\parallel (1 - v_\parallel^2/c^2)} \frac{\partial F_e}{\partial v_\parallel} = \text{const.},$$

they make no contribution to electron diffusion, i.e., they do not interact with the plasma at all. For this reason, the spectrum may be determined only to within such a packet of non-interacting "extraneous" waves. The energy minimality requirement excludes this packet. In order to fulfill this requirement, a constant positive term

$$\frac{v_a}{c} \int F_e'(v_\parallel)\, dv_\parallel = -\alpha \frac{\sqrt{1-x^2}}{x},$$

must be added to the function

$$c' = \alpha \frac{\sqrt{1-x_0^2}}{x_0}, \quad x_0 \cong \sqrt{T_i/T_e}$$

where $\alpha = (2/\pi)(v_a/c)(m/M)(\nu_i/\omega_i)$; this latter function increases monotonically in the region under consideration.

Hence, the spectrum of ionic-acoustic oscillations may be written as

$$\frac{\varepsilon_k/R_D}{NT_e} = 2\left(\frac{m}{M}\right)^{3/2} \frac{\nu_e}{\omega_e} \times \left\{\frac{\sqrt{1-x_0^2}}{x_0} - \frac{\sqrt{1-x^2}}{x}\right\} \bar{x} \cdot x^3 (1-x^2)^{3/2}. \quad (48)$$

$$\bar{x} \equiv v_a/c.$$

As we have already noted, ionic sound is present only when $T_i/T_e \ll 1$, i.e., when $x_o \ll 1$. Thus the spectral density is given by the formula

$$\frac{\varepsilon_k/R_D}{NT_e} \simeq \left(\frac{m}{M}\right)^{3/2} \frac{T_e}{T_i} (\omega_i \tau_i)\, \bar{x} \cdot x^3 (1-x^2)^{3/2} \quad (\tau_i \equiv 1/\nu_i) \quad (49)$$

throughout almost the entire frequency region. The total electrostatic energy density of the ionic-acoustic oscillations is equal to

$$\frac{E^2}{8\pi N T_e} \sim \frac{1}{60\pi^3} \frac{T_e}{T_i} (\omega_i \tau_i) \overline{x}. \tag{50}$$

Formula (49) can be used to find the energy distribution in the low-frequency portion of the spectrum (usually measured experimentally):

$$\varepsilon_\omega = \frac{\varepsilon_k}{d\omega/dk} \approx \frac{\varepsilon_k}{c} = \frac{N T_e}{\omega_i} \left(\frac{m}{M}\right)^{3/2} \frac{T_e}{T_i} (\omega_i \tau_i) \overline{x} \left(\frac{\omega}{\omega_i}\right)^3. \tag{51}$$

The effect of these oscillations on transfer processes in a homogeneous plasma is small, since:

1. The spectrum is one-dimensional ($E_k \| H$), so that the electric fields of the oscillations cannot result in the appearance of "anomalous diffusion" of particles across the magnetic field.
2. The oscillations "occupy" but a small region of the velocity space /474 ($c/(T_e/m)^{1/2} \simeq \sqrt{m/M}$). From the standpoint of electric current flow the electrons can be broken down into two groups: the resonance electrons retarded by the ionic oscillations, and the non-resonance electrons retarded in colliding with neutral particles. Since the second group contains $\sqrt{M/m}$ as many electrons, the effect of the additional "collective" resistance in this case is small.

6. Effect of Cyclotron Waves on the Lifetime of Particles in a Trap With Magnetic Mirrors

The appearance of instability and noises in traps with magnetic mirrors[1] may be due to the anisotropy of the particle distribution function in the velocity space. This anisotropy can be the result of the corresponding injection or heating of the plasma, but it can also be inherent in the method of confine-

[1]From now on, we assume in all cases that the magnetohydrodynamic instability of such systems has been eliminated by appropriate stabilizing measures.

ment: as the particles approach their turning points, their longitudinal energy is completely transformed into trasverse energy.

As we know, such anisotropy results in the buildup in the plasma of a circularly polarized electromagnetic wave [3]. The phase velocity of the wave is high (on the order of the Alfvén velocity). Hence, in order for the velocity distribution of the particles to contain resonance particles for which the wave frequency is equal to the Larmor frequency by virtue of the Doppler effect, it is necessary that

$$v_{\parallel} = \frac{\omega \pm \omega_H}{K_{\parallel}} \sim \sqrt{\frac{T}{m}};$$

since $\omega/k_{\parallel} \sim H\sqrt{4\pi NM} > \sqrt{T/M}$, this is possible only when the minus sign is chosen. This sign corresponds to a wave circularly polarized in the direction of particle rotation.

Let us consider the mean effect of such a wave on particle motion in a spatially homogeneous plasma. For simplicity, we limit ourselves to the case of purely longitudinal propagation. Under the action of the rotating electric

$$E_{-} = E_x - iE_y \tag{52}$$

and magnetic

$$H_{-} = -\frac{ic}{\omega} kE_{-} = H_x - iH_y \tag{53}$$

fields of the wave with the wave number k, the change in the particle distribution function f_k in a spatially homogeneous plasma is equal to

$$f_k = iE_{-k}\frac{1}{\omega_k}\frac{(\omega_k - kv_{\parallel})(\partial f_0/\partial v_{\perp}) + kv_{\perp}(\partial f_0/\partial v_{\parallel})}{\omega - kv_{\parallel} - \omega_{H\alpha}}\frac{e_\alpha}{2m_\alpha}e^{i\varphi} \tag{54}$$

where f_0 is the function of particle distribution by velocities averaged over many oscillations, $v_{\parallel}$, $v_{\perp}$, and ϕ are the cylindrical coordinates in the velocity space, e_{α} and m_{α} are the charge and mass of the particles.

The equation for the function f_0 is obtained by averaging the Boltzmann equation over small oscillations to yield

$$\frac{df_0}{dt} = \left\langle \sum_{kk'} \left[\left(\mathbf{E}_k + \frac{\mathbf{v} \times \mathbf{H}_k}{c}\right) e^{-i\omega_k t + ikz} + \text{c.c.}\right] \times \left[\frac{\partial f_{k'}}{\partial \mathbf{v}} e^{-i\omega_k t + ik'z} + \text{c.c.}\right] \right\rangle . \tag{55}$$

This leads to the equation

$$\frac{df_0}{dt} = \left\{\left(1 - \frac{kv_{\parallel}}{\omega_k}\right) \frac{1}{v_{\perp}} \frac{\partial}{\partial v_{\perp}} v_{\perp} + v_{\perp} \frac{\partial}{\partial v_{\parallel}} \frac{k}{\omega_k}\right\} D_H \times \left\{\left(1 - \frac{kv_{\parallel}}{\omega_k}\right) \frac{\partial}{\partial v_{\perp}} + \frac{kv_{\perp}}{\omega_k} \frac{\partial}{\partial v_{\parallel}}\right\} f_0 , \tag{55a}$$

where

$$D_H = \frac{1}{4} \frac{|E_k|^2}{|d\omega/dk - v_{\parallel}|} \frac{e_{\alpha}^2}{m_{\alpha}^2} ,$$

and the following relationship exists between $v_{\parallel}$ and ω_k:

$$v_{\parallel} = \frac{\omega_k - \omega_{H\alpha}}{k} .$$

Thus, the action of the waves alters both the longitudinal and transverse velocities of the particles, i.e., the magnetic moment $\mu_{\alpha} = m_{\alpha} v_{\perp}^2/2H$. Equation (55) has the form of the Fokker-Planck equation in the velocity space. As a result of "diffusion" the distribution function is smoothed along the lines

$$v_{\parallel} - \frac{v_{\perp}^2 k}{2\omega_{H\alpha}} = \text{const}.$$

Since we are considering small-amplitude waves for which energy transfer from one Fourier harmonic to another is negligibly small as compared with the flux of energy from the particles, the change in the wave energy or in the "coefficient of diffusion" D_H proportional to it is determined as in the ordinary linear theory:

$$\frac{d \ln D_H}{dt} = 2\gamma \{f_0\}, \tag{56}$$

where $\gamma \equiv \mathrm{Im}\,\omega_k$ can in turn be found from the dispersion equation for small oscillations:

$$\frac{k^2c^2}{\omega^2} - 1 = \frac{1}{2}\sum_{\alpha} \frac{\omega_\alpha^2}{\omega^2}\int \frac{(\omega - kv_\parallel)(\partial f_0/\partial v_\perp) + kv_\perp(\partial f_0/\partial v_\parallel)}{\omega - \omega_{H\alpha} - kv_\parallel}\, v_\perp\, d\mathbf{v}. \tag{57}$$

Here

$$\omega_\alpha^2 = 4\pi N e_\alpha^2/m_\alpha, \quad \omega_{H\alpha} = e_\alpha H/m_\alpha c.$$

The instability just described develops much more rapidly on the electron branch of oscillations. The electrons are confined by the space charge of the ions, however. Hence, plasma escape is in the final analysis determined by the escape of ions. If the wavelength is considerably smaller than the system dimensions $kL \gg 1$, then the change in wave energy can be computed using an approximation borrowed from the field of geometric optics. In this case the change in the spectral density ε_k of the oscillations at a given point is determined by the equation

/475

$$\frac{\partial \varepsilon_k}{\partial t} + \frac{\partial \omega_k}{\partial k}\frac{\partial \varepsilon_k}{\partial z} - \frac{\partial \omega_k}{\partial z}\frac{\partial \varepsilon_k}{\partial k} = 2\gamma\{f_0\}\varepsilon_k + J_k(z), \tag{58}$$

where $J_k(z)$ is the power of thermal noise sources for the given harmonic k as determined from Kirchhoff's law. The third term in the left side of (58) can generally be neglected. We thus have the following equation for the change in the coefficient of diffusion in space:

$$\frac{\partial D_H}{\partial t} + \frac{\partial \omega_k}{\partial k}\cdot\frac{\partial D_H}{\partial z} = 2\gamma D_H + \dot{D}_{H_0}. \tag{59}$$

(D_{H_o} is expressed in terms of the equilibrium fluctuations of the electric field).

In the same approximation kL >> 1 the diffusion of the particles under the action of waves is described by equation (55). If the field varies weakly for dimensions on the order of those of the ionic orbit, then the drift approximation yields the following expression for the mean distribution function f_o:

$$\frac{\partial f_0}{\partial t}+v_\parallel\frac{\partial f_0}{\partial z}-\frac{\mu_\alpha}{m_\alpha}\frac{\partial H_0}{\partial z}\frac{\partial f_0}{\partial v_\parallel}$$
$$=\left\{\frac{1}{H}\frac{\partial}{\partial\mu_\alpha}+\frac{\partial}{\partial v_\parallel}\frac{k}{\omega_{H\alpha}}\right\}\frac{2\mu}{H}D_H\left\{\frac{\partial}{\partial\mu_\alpha}+\frac{kH}{\omega_{H\alpha}}\frac{\partial}{\partial v_\parallel}\right\}f_0. \quad (60)$$

In principle, equations (57), (59), and (60) permit complete solution of the problem of the "anomalous" escape of particles from a trap. We, however, will limit ourselves to computing the coefficient of diffusion D_H and the particle escape time.

In the absence of particle escape, the function of particle distribution by velocities in a field varying up to the magnitude H_m in the "mirror" is given by the expression

$$f_0=f_m(v_\parallel,v_\perp)\,\varepsilon\left(v_\perp\frac{H_m-H}{H}-v_\parallel^2\right)$$
$$\varepsilon(x)=\begin{cases}1 & x>0\\ 0 & x<0\end{cases} \quad (61)$$

where $f_m(v_\parallel,\ v_\perp)$ is the Maxwellian distribution function. Further, let us suppose that this function is isotropic. With such a distribution (with an unfilled cone of escaped particles), equation (57) implies that

$$\gamma=-\sqrt{\pi}\frac{k}{\sqrt{2T/M}}v_\parallel^2e^{-Mv_\parallel^2/2T}$$
$$\left(1+\frac{H_m+H}{H_m-H}\frac{k}{\omega_{Hi}}\frac{v_\parallel^3}{(2T/M)^{3/2}}\right) \quad (62)$$

and

$$k^3 = -\frac{\omega H_i^2}{v_\parallel v_A^2}\sqrt{\frac{H_m - H}{H_m}}; \quad v_A = \frac{H_m}{\sqrt{4\pi NM}}. \tag{63}$$

In these expressions $v_\parallel = (\omega - \omega_{Hi})/k$, i.e., the velocity of the particles in resonance with a wave having the given wave vector k. The direction of the z-axis is so chosen that the magnetic field increases as $z \to \infty$. Expression (63) indicates that the rising wave penetrates from the mirror into the interior of the trap.

As a result of wave buildup, the coefficient of diffusion D_H likewise increases with distance from the mirror:

$$\ln(D_H/D_{H_0}) \simeq 2\int_0^z \frac{\gamma(z)}{\partial\omega/\partial k}\,dz = \int_0^z \gamma(z)\frac{dz}{v_\parallel}. \tag{64}$$

The quantity $\phi = \int_0^z [\gamma(z)/v_\parallel]dz$ is maximum for waves for which

$$v_\parallel \sim \sqrt{\frac{2T}{M}}\beta^{1/2}\left(\frac{14}{3}\right)^{3/2} \qquad \text{with } z = 2\beta L$$

The coefficient of diffusion thus turns out to be equal to

$$\ln(D_H/D_{H_0}) \approx \frac{L}{r_{\Lambda i}}\beta^{4/3},$$

in order of magnitude. $r_{\Lambda i}$ in the latter expression is the radius of Larmor rotation of the ion. Since the diffusion associated with thermal oscillations is close to the diffusion due to pair collisions, the quantity $L/r_{\Lambda i}\beta^{4/3}$ serves as a rough approximation of the reduction of particle lifetimes in the trap.

In existing traps this quantity is very small, since their $\beta \sim 10^{-4} - 10^{-5}$, and $z_{\Lambda i} \sim 1 - 0.1L$.

The quantity $\beta_c \sim (r_{\Lambda i}/L)^{3/4}$ gives the critical value of β above which anomalous escape occurs even in mirror-equipped traps stabilized against magnetohydrodynamic instability.

We note that any increase in the degree of anisotropy of the distribution function resulting from injection or magnetic compression increases the diffusion we have been describing.

7. Conclusion

The quasilinear approximation is a useful method of describing weakly nonlinear processes in plasmas. Full description of the steady-state of a plasma situated in external fields or of the relaxation process in an unstable plasma, in addition to consideration of the reaction effect of oscillations on the particle distribution, also requires knowledge of the specific dissipation processes (e.g., of the "trajectory intersection" type), as well as the laws of disintegration of the spectrum of plasma oscillations. At the present time these processes have been studied only for certain special and simple models of the one-dimensional Langmuir electron oscillation type. Substantial progress in this field is apparently attainable by way of "computer experiments".

REFERENCES

1. Davydov, B. I.: Paper in "Plasma Physics and the Problem of Controlled Thermonuclear Reactions" ("Fizika Plazmy i Problema Upravlyayemykh Termoyadernykh Reaktsiy"), Academy of Sciences USSR, Vol. I, p. 77, Moscow, 1958.

2. Vedenov, A. A., Ye. P. Velikhov and R. Z. Sagdeyev: Yadernyy Sintez, Vol. I, pp. 82-100, 1961.

3. Vedenov, A. A., Ye. P. Velikhov and R. Z. Sagdeyev: Uspekhi Fizicheskikh Nauk, Vol. 73, p. 701, 1961.

Translated for the National Aeronautics and Space Administration by the FRANK C. FARNHAM COMPANY.